SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 13

MATH 241

12.3.8

Proof. When $-2 < x < 0$, since $0 < -x < 2$, we should use the formula in the definition on $[0,2)$, so $f(-x) = -(-x) + 5 = x + 5 = f(x)$. Similarly when $0 \leq x < 2$, $f(-x) = (-x) + 5 = -x + 5 = f(x)$. Therefore the function is even. □

12.3.13

Proof. This is an even function, so we only have the Fourier cosine series. $b_n = \frac{2}{\pi}\int_0^\pi x \cos nx dx$, then integrate by parts. This is just showing the example on how to deal with absolute value function. □

12.3.23

Proof. This is an even function, so we only have the Fourier cosine series. $b_n = \frac{2}{\pi}\int_0^\pi \sin x \cos nx dx$, then use the trignometry formula. □

12.3.29

Proof. Fourier cosine series: $a_0 = \frac{2}{\pi}[\int_0^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}(\pi - x)dx] = \frac{2}{\pi}[\frac{1}{2}x^2|_0^{\frac{\pi}{2}} - \frac{1}{2}(\pi - x)^2|_{\frac{\pi}{2}}^{\pi}] = \frac{2}{\pi}[\frac{\pi^2}{8} + \frac{\pi^2}{8}] = \frac{1}{2}\pi$.

$$a_n = \frac{2}{\pi}[\int_0^{\frac{\pi}{2}} x \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\cos nx dx]$$

Here you may calculate each of them, but if you change of variable $x \mapsto \pi - x$ in the second integral, you will get the second term changed into $\int_{\frac{\pi}{2}}^0 x \cos n(\pi - x)(-dx) = \int_0^{\frac{\pi}{2}} x \cos n(\pi - x) dx$. But what is $\cos n(\pi - x)$? By periodicity, it only depends on the parity of n. When n is even, this is $\cos(-nx) = \cos nx$, when n is odd it is $\cos(\pi - nx) = -\cos nx$. So we can write it as $(-1)^n \cos nx$. Thus we changed the second term into $(-1)^n \int_0^{\frac{\pi}{2}} x \cos nx dx$. Therefore when n is odd it is the opposite of the first term, so you end up with zero. When $n = 2k$ we reduce to

$$
\begin{aligned}
a_n &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\cos 2kx dx \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} xd(\frac{\sin 2kx}{2k}) \\
&= \frac{4}{\pi}[\frac{1}{2k}(x\sin 2kx)|_0^{\frac{\pi}{2}} - \frac{1}{2k}\int_0^{\frac{\pi}{2}} \sin 2kx dx] \\
&= -\frac{4}{\pi}\cdot\frac{1}{2k}\cdot(-\frac{\cos 2kx}{2k})|_0^{\frac{\pi}{2}} \\
&= -\frac{1}{k^2\pi}[\cos k\pi - \cos 0] \\
&= -\frac{1}{k^2\pi}[(-1)^k - 1]
\end{aligned}
$$

Therefore the answer can be written as

$$f(x) = \frac{1}{4}\pi + \sum_{n=1,n=2k}(-\frac{(-1)^k-1}{k^2\pi})\cos nx = \pi + \sum_{k=1}(-\frac{(-1)^k-1}{k^2\pi})\cos 2kx$$

So if you have made any substitution for n, do not forget to plug it back at the end, and also change the index. If you observe the answer, when k is even it is zero, so you can again let $k = 2l + 1$, and the answer can be written as

$$f(x) = \frac{1}{4}\pi + \sum_{l=1}^{\infty}\frac{2}{(2l+1)^2\pi}\cos(4l+2)x$$

The appearance is different from the answer key, but you can check they are actually equal, and our answer here looks more succint.

For Fourier sine series, you can try the same trick. Hint: $\sin(n(\pi - x)) = (-1)^{n+1}\sin x$. Why? □

12.3.35

Proof. This is the example of translation extension of a function defined on $[0, 2\pi]$, so the period should be 2π, and the symmetric interval we are realling thinking about is $[-\pi, \pi]$.

Therefore $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx dx$, and $b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx dx$. Then integrate by parts. So the way to memorize these formulae is to think of $p = \pi = \frac{2\pi}{2}$, and use the usual formula for Fourier series on $[-\pi, \pi]$ except for the integral limit is from 0 to 2π. Although you may not need to memorize them at all...

Note: Even it is an even function, you cannot just calculate the cosine series! Because we are really looking at its extension by translation, and that function, if you draw the graph, is not even or odd. □

12.4.2

Proof. In this example, the function is defined on $(0, 2)$, so we need to extend it by translation. **Note**: there is no translation by reflection if you want a complex series. Therefore the period is 2, so $p = 1$.

$$c_n = \frac{1}{2 \cdot 1} \int_0^2 f(x) e^{in\pi x} dx = \frac{1}{2} \int_1^2 e^{in\pi x} dx = \frac{1}{2} \frac{1}{in} [e^{2in\pi} - e^{in\pi}] = \frac{1}{2in} [1 - (-1)^n]$$

This is only true for $n =\neq 0$. When $n = 0$ you get

$$c_0 = \frac{1}{2 \cdot 1} \int_0^2 f(x) dx = \frac{1}{2}$$

Therefore $f(x) = \frac{1}{2} + \sum_{n \neq 0} \frac{1}{2in} [1 - (-1)^n] e^{-in\pi x}$. **Note** the index you want to exclude $n = 0$ and also pay attention to the sign!(especially me, I guess) □

12.4.10

Proof. Although it is asking for Fourier complex series, it is easier to calculate in usual Fourier series and notice $c_n = \frac{1}{2}(a_n - ib_n)$ when $n \neq 0$, so $|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$, and $c_0 = \frac{1}{2} a_0$, so $|c_n| = \frac{1}{2} |a_0|$. This is on $(0, \pi)$, so by translation extension $p = \frac{\pi}{2}$, and $\omega = \frac{\pi}{p} = 2$.

$$a_0 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos x dx = \frac{2}{\pi}$$

So $|c_0| = \frac{1}{\pi}$.

$$a_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos x \cos 2nx dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos(1 + 2n)x + \cos(1 - 2n)x dx = \frac{1}{\pi} [\frac{\sin(\frac{2n+1}{2}\pi)}{1 + 2n} + \frac{\sin(\frac{-2n+1}{2}\pi)}{1 - 2n}]$$

Note that $\sin(\frac{2n+1}{2}\pi) = (-1)^n$, therefore $a_n = \frac{1}{\pi}[\frac{(-1)^n}{1+2n} + \frac{(-1)^{-n}}{1-2n}] = \frac{2(-1)^n}{\pi(4n^2-1)}$.

Similarly you can use the formula $\sin 2nx \cos x = \frac{\sin(2n+1)x + \sin(2n-1)x}{2}$ to calculate $b_n = \frac{4n}{\pi(4n^2-1)}$. Therefore $|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \frac{\sqrt{4n^2+1}}{4n^2-1}$. In general, when you calculate the frequency spectrum, **Do not forget** to take the absolute value[1] But in our example $4n^2 - 1$ does no need an absolute value since we are assuming $n \neq 0$ therefore $4n^2 - 1$ is always positive. Do remember that the frequency spectrum's second coordinate $|c_n|$ is a nonnegative real number, so double check your answer and see whether it is even possible to be negative.

Therefore the frequency spectrum is $(2n, \frac{\sqrt{4n^2+1}}{4n^2-1})$ when $n \neq 0$, and $(0, \frac{1}{\pi})$ at $n = 0$. **Note**: always check whether 0 needs special care! □

Fall 11, #2

Proof. That is the evaluation of the Fourier series at $x = 0$, so it converges to $\frac{f(0+)+f(0-)}{2} = \frac{3}{2}$. □

Fall 11, #5

[1]Sometimes even for n, Because n can be a negative number now.

Proof. That is an even function, so $b_k = 0$, therefore $\sum a_k^2 b_k^2 = 0$ □

Fall 11, #14

This problem part (b) is changed into the following form:

What is the value for

$$\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \cdots$$

So the order of the signs are changed.

Proof. Since it is an even function, we will get a Fourier cosine series.

$$a_0 = \frac{2}{\pi}\int_0^\pi x dx = \frac{1}{\pi}[\pi^2 - 0] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx dx = -\frac{2}{n\pi}\int_0^\pi \sin nx dx = \frac{2}{n^2\pi}[(-1)^n - 1]$$

Therefore we get the Fourier cosine series:

$$f(x) = \frac{1}{2}\pi + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}[(-1)^n - 1]\cos nx$$

For this series, only when n is odd we have a nonzero coefficient. So let $n = 2k+1$, then notice the limits for k is from 0 to ∞, instead of from 1 to ∞.

$$f(x) = \pi + \sum_{k=1}^{\infty} -\frac{4}{(2k+1)^2\pi}\cos(2k+1)x$$

If we only look at the coefficients, we should expect a series with only odd reciprocals. But that is still not we need in (b). We want the signs to change in a period of 4. The trick is to let $\cos(2k+1)x$ come in as well. Let $x = 0, \pi$ will not work, since for those values $\cos(2k+1)x = 1$ or -1 does not change signs. However, if you try $x = \frac{1}{2}\pi$, you will get $\cos\frac{2k+1}{2}\pi = 0$, it still does not work. But you can let $x = \frac{\pi}{4}$, then you have $\cos\frac{2k+1}{4}\pi$, you can try the first k's and discover that $\cos\frac{2k+1}{4}\pi = \begin{cases} \frac{\sqrt{2}}{2} & k = 4l \\ -\frac{\sqrt{2}}{2} & k = 4l+1 \\ -\frac{\sqrt{2}}{2} & k = 4l+2 \\ \frac{\sqrt{2}}{2} & k = 4l+3 \end{cases}$ so they all differ only up to signs, and the signs change in period equal to 4, in the type as $+, -, -, +$. That is why we say the problem was originally designed wrong, and it should be changed into

$$\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \cdots$$

this type. Therefore we can just plug in $x = \frac{\pi}{4}$, $f(x) = \frac{\pi}{4}$, on the Fourier side we have $\frac{1}{2}\pi - \frac{4}{\pi^2}\frac{\sqrt{2}}{2}[\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \cdots]$. This will solve out

$$\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \cdots = \frac{\sqrt{2}\pi^3}{16}$$

□

Fall 08, #8

Proof. Without integration at all, we can change $\sin^4 x = (\frac{1}{2}[1 - \cos 2x])^2$ by double angle formula, then $\frac{1}{4}[1 - 2\cos 2x + \cos^2 2x]$ is still not in a standard form for a Fourier series, the trouble is $\cos^2 2x$, so you need to use double angle formula again get $\frac{1}{4}[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)] = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$ □